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Strictly regular ternary Hermitian forms

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ABSTRACT

A positive definite integral Hermitian form is called strictly regular if it primitively represents all integers that can be primitively represented locally everywhere by the form itself. In this article, we show that there are only finitely many equivalence classes of primitive strictly regular positive definite integral ternary Hermitian forms over a fixed imaginary quadratic field.

1. Introduction

A main question in the study of positive definite integral quadratic forms is determining which integers are represented by a given form. Specifically, given a positive definite integral quadratic form, Q, for which positive integers a are there integers $x_1, \dots, x_n$ such that $Q(x_1, \dots, x_n) = a$? In order for such a representation to exist it is necessary that there are local representations over the p-adic completion $\mathbb{Z}_p$ of $\mathbb{Z}$ for every prime p;

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however, this is not sufficient. When we do have sufficiency we call the forms *regular*. Regular quadratic forms were first studied by Dickson in [2].

It is known that primitive regular positive definite integral ternary quadratic forms lie in only finitely many equivalence classes, but the same is not true for primitive regular positive definite integral quaternary quadratic forms [4]. In the quaternary case, Earnest, Kim and Meyer [6] are able to achieve finiteness by restricting their attention to primitive strictly regular positive definite integral quaternary quadratic forms. A positive definite integral quadratic form Q is said to be *strictly regular* if for every positive integer a there exists a primitive representation by Q over $\mathbb{Z}$ whenever there is a primitive representation by Q over $\mathbb{Z}_p$ for every prime p.

There are similar finiteness results for regular positive definite integral Hermitian forms. Earnest and Khosravani [5] show that there are only finitely many classes of primitive regular positive definite integral binary Hermitian forms over a fixed imaginary quadratic field. More precisely, they deal with the more general notion of Hermitian lattices and show that for a positive definite integral binary Hermitian lattice L, the cardinality of $\mathcal{E}(L)$ tends to infinity as the volume of L tends to infinity, where $\mathcal{E}(L)$ is the set of integers represented by L locally everywhere but not represented by L itself. Chan and Rokicki [1] show that for a fixed totally real number field F of odd degree over $\mathbb{Q}$, there are only finitely many CM extensions E/F for which there exists a regular normal positive definite integral Hermitian lattice over the ring of integers of E. In particular, they have shown that a regular normal positive definite integral binary Hermitian lattice exists over the field $\mathbb{Q}(\sqrt{-m})$ if and only if m is

$$1, 2, 3, 5, 6, 7, 10, 11, 15, 19, 23 \text{ or } 31.$$

Similarly, Kim, Kim and Park [11] find that a primitive regular subnormal positive definite integral binary Hermitian lattice with $\mathfrak{n}L = 2\mathcal{O}$ exists over the field $\mathbb{Q}(\sqrt{-m})$ if and only if m is

$$1, 2, 5, 6, 10, 13, 14, 17, 21, 22, 29, 34, 37 \text{ or } 38.$$

Furthermore, in [12], they show that primitive regular subnormal positive definite integral binary Hermitian lattices with $\mathfrak{n}L = m\mathcal{O}$ appear over infinitely many imaginary quadratic fields $\mathbb{Q}(\sqrt{-m})$.

Our first result concerns the finiteness of primitive regular positive definite integral ternary Hermitian lattices.

Theorem 1.1. *An imaginary quadratic field supports a primitive regular positive definite integral binary Hermitian lattice if and only if it supports infinitely many isometry classes of primitive regular positive definite integral ternary Hermitian lattices.*

We will then concentrate our attention on strict regularity, and the main objective of this paper is to prove the following:

Theorem 1.2. *Over a fixed imaginary quadratic field, there are only finitely many isometry classes of primitive positive definite integral ternary Hermitian lattices that are strictly regular.*

2. Notations and preliminaries

Throughout this paper we will adopt the terminology and notation for lattices as in [13]. For background results and terminology specific to the Hermitian case, the reader may refer to the papers [7,10] and [14].

We fix once for all an imaginary quadratic field $E = \mathbb{Q}(\sqrt{-m})$, where m is a squarefree positive integer. Let $\mathcal{O}$ be the ring of integers of E. We have $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega := \omega_m = \sqrt{-m}$ if $m \equiv 1, 2 \pmod 4$ and $\omega := \omega_m = (1 + \sqrt{-m})/2$ if $m \equiv 3 \pmod 4$. For any $a \in E$, let $\overline{a}$ denote the complex conjugation of a, and we define the *norm* and *trace* of a by $\mathbf{N}(a) = a\overline{a}$ and $\mathbf{Tr}(a) = a + \overline{a}$ respectively.

A Hermitian space (V, h) is a vector space over $\mathbb{Q}(\sqrt{-m})$ with a Hermitian map $h : V \times V \to \mathbb{Q}(\sqrt{-m})$ satisfying the following conditions:

1. $h(v, w) = \overline{h(w, v)}$ for any $v, w \in V$,
2. $h(v_1 + v_2, w) = h(v_1, w) + h(v_2, w)$ for any $v_1, v_2, w \in V$,
3. $h(\alpha v, w) = \alpha h(v, w)$ for any $\alpha \in E$ and $v, w \in V$.

We will write $h(v) = h(v, v)$ for brevity. From condition (1), we know that $h(v) = \overline{h(v)}$ and hence $h(v) \in \mathbb{Q}$ for all $v \in V$. The *unitary group* of the space (V, h) is the set of all bijective E-linear maps $\sigma : V \longrightarrow V$ which satisfy $h(\sigma(x), \sigma(y)) = h(x, y)$ for all $x, y \in V$ and it is denoted by $U(V)$.

A Hermitian lattice L is defined as a finitely generated $\mathcal{O}$-module in the Hermitian space V. In particular, we say L is a lattice on V if $EL = V$. We will assume throughout that all Hermitian lattices L are *positive definite integral* in the sense that $h(v) \in \mathbb{Z}^+$ for all nonzero element $v \in L$. If $a = h(v)$ for some primitive vector v of L, then we say a is *primitively* represented by L, denoted $a \xrightarrow{*} L$, otherwise $a \stackrel{*}{\not\longrightarrow} L$. And L is said to be (*strictly*) *universal* if it (primitively) represents every positive integer. Assuming that M and L are lattices in the Hermitian space V, we say that M is primitively represented by L, written $M \xrightarrow{*} L$, if there is some $\sigma \in U(V)$ such that $\sigma(M)$ is a direct summand of L. Note that for a free unary Hermitian lattice $M = \mathcal{O}v$ with $h(v) = a$, $M \xrightarrow{*} L$ if and only if $a \xrightarrow{*} L$. All of these definitions can be given analogously in the local case.

For any rational prime p, the localization of E and $\mathcal{O}$ at p are defined as $E_p := E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{O}_p := \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, respectively. Similarly, for a non-degenerate Hermitian space V over E, we put $V_p := V \otimes_E E_p$. If L is a lattice on V, the localization of L at p is the $\mathcal{O}_p$-lattice $L_p := L \otimes_{\mathcal{O}} \mathcal{O}_p$. A prime p of $\mathbb{Q}$ is said to be ramified, split or inert, if it behaves accordingly in the extension $E/\mathbb{Q}$. When p splits in E, then $E_p \cong \mathbb{Q}_p \times \mathbb{Q}_p$ with $\mathbb{Q}_p$ embedded diagonally. When p does not split in E, then E_p is a quadratic extension of $\mathbb{Q}_p$.

Let L be a lattice on an n-dimensional non-degenerate Hermitian space V over E. There exist vectors $v_1, \dots, v_n$ in V and fractional $\mathcal{O}$-ideals $\mathfrak{U}_1, \dots, \mathfrak{U}_n$ such that $L = \mathfrak{U}_1 v_1 + \cdots + \mathfrak{U}_n v_n$. The *volume* of L, denoted $\mathfrak{v}L$, is defined to be the fractional $\mathcal{O}$-ideal

$$\mathfrak{v}L = \left(\mathfrak{U}_1\overline{\mathfrak{U}_1}\right)\cdots\left(\mathfrak{U}_n\overline{\mathfrak{U}_n}\right)\det\left(h(v_i, v_j)\right).$$

Note that each of the $\mathfrak{U}_i$'s can be written as a product of integral powers of prime ideals of $\mathcal{O}$. For any prime ideal $\mathfrak{P}$ of $\mathcal{O}$, let p be the rational prime which lies below $\mathfrak{P}$. Then we have

$$\mathfrak{P}\overline{\mathfrak{P}} = \begin{cases} p^2\mathcal{O}, & \text{if } p \text{ is inert in } E; \\ p\mathcal{O}, & \text{otherwise.} \end{cases}$$

Consequently, there is a unique positive rational number δ_L with the property that $\mathfrak{v}L = \delta_L\mathcal{O}$, and we refer to δ_L as the discriminant of L. For $j = 1, \dots, n$, the jth successive minimum of L is defined to be the smallest positive integer $\mu_j(L)$ with the property that

$$\dim\left(\text{span}_E\{v \in L : h(v) \le \mu_j(L)\}\right) \ge j.$$

The existence of linearly independent vectors $x_1, \dots, x_n \in L$ such that $h(x_j) = \mu_j(L)$ can be established in the same way as in the case of quadratic $\mathbb{Z}$-lattices; see [3, Lemma 2.2]. For $m < n$, we refer to the sublattice $M = \text{span}_E\{x_1, \dots, x_m\} \cap L$ as a leading m-ary sublattice of L. Note that when M is a leading m-ary sublattice of L, M is a primitive sublattice of L and $\mu_j(M) = \mu_j(L)$ for $1 \le j \le m$. By [5, Proposition 1.1], one has $\delta_L \le \mu_1(L) \cdots \mu_n(L)$.

For a Hermitian $\mathcal{O}$-lattice (L, h) of rank n, we can always associate it with a quadratic $\mathbb{Z}$-lattice $(L_{\mathbb{Z}}, b_h)$ in the following way: (1) $L_{\mathbb{Z}} = L$ setwise but viewed as a lattice over $\mathbb{Z}$; (2) $b_h = \mathbf{Tr} \circ h$. A straightforward calculation shows that $\mathfrak{v}L_{\mathbb{Z}}\mathcal{O} = (-D_E)^n \mathfrak{v}L^2$ where D_E is the discriminant of E. In particular, when L is a binary Hermitian lattice, the discriminant of $L_{\mathbb{Z}}$ is a square.

The *norm* of L, denoted $\mathfrak{n}L$, is the fractional $\mathcal{O}$-ideal generated by $h(L)$. The *scale* of L is the fractional $\mathcal{O}$-ideal $\mathfrak{s}L$ generated by the set $\{h(x, y) : x, y \in L\}$. It is clear that $\mathfrak{n}L \subseteq \mathfrak{s}L$. If $\mathfrak{n}L = \mathfrak{s}L$, we call L *normal*, otherwise *subnormal*. All of these definitions carry over to $\mathcal{O}_p$-lattices. It is known that subnormal $\mathcal{O}_p$-lattices exist only when p is ramified in E. We say that L is *primitive* in the sense that $\mathfrak{n}L = \mathcal{O}$.

3. Local preliminaries

Lemma 3.1. *Let K be a positive definite quaternary quadratic $\mathbb{Z}$-lattice with square discriminant. Then there exists a prime p such that K_p is anisotropic.*

Proof. Let $U = \mathbb{Q}K$. Suppose that U is isotropic at all finite spots. Then $S_p(U_p) = 1$ when p is odd and $S_2(U_2) = -1$. Since $S_\infty(U_\infty) = 1$, it follows that

$$S_\infty(U_\infty) \prod_p S_p(U_p) = -1.$$

This contradicts Hilbert's Reciprocity Law. Hence there is a prime p such that U_p is anisotropic, i.e., K_p is anisotropic. □

Lemma 3.2. *Let K_p be an anisotropic quadratic $\mathbb{Z}_p$-lattice. Then there is a positive integer $e = e(K_p)$ such that K_p does not primitively represent any p-adic integer in $p^e\mathbb{Z}_p$.*

Proof. The set $P(K_p) := K_p \backslash pK_p$ is a compact subset of K_p. Since the quadratic map $Q : K_p \to \mathbb{Z}_p$ is continuous, $Q(P(K_p))$ is also compact. Suppose that for any positive integer m, there is some element $v_m \in P(K_p)$ such that $Q(v_m) \in p^m\mathbb{Z}_p$. Then $\{Q(v_m)\}_{m \geq 0}$ is a Cauchy sequence with limit 0. Since $Q(P(K_p))$ is compact, $0 \in Q(P(K_p))$, it follows that K_p is isotropic. This is a contradiction. □

Lemma 3.3. *Let K be a positive definite binary Hermitian $\mathcal{O}$-lattice. Then there exists a prime p and a positive integer $f = f(K, p)$ such that K_p is anisotropic and K_p does not primitively represent any p-adic integer in $p^f\mathbb{Z}_p$.*

Proof. Let $K_\mathbb{Z}$ be the associated quaternary quadratic $\mathbb{Z}$-lattice of K. Then, for any prime p, K_p is anisotropic if and only if $(K_\mathbb{Z})_p$ is anisotropic, and $a \overset{*}{\not\to} (K_\mathbb{Z})_p$ implies that $\frac{1}{2}a \overset{*}{\not\to} K_p$. Since $K_\mathbb{Z}$ has square discriminant, the proof follows from Lemmas 3.1 and 3.2 immediately. □

Lemma 3.4. *Suppose L_p is a Hermitian $\mathcal{O}_p$-lattice such that $L_p = J_p \perp K_p$ where J_p is universal and $K_p \neq 0$. Then L_p is strictly universal.*

Proof. Take a primitive vector $x \in K_p$, and let $a = h(x)$. For any element $d \in \mathbb{Z}_p$, since J_p is universal, there is some element $y \in J_p$ such that $h(y) = d - a$. Therefore $h(y + x) = d$ where $y + x$ is a primitive vector in L_p. Hence L_p is strictly universal. □

4. Watson's transformations

The following definitions are the Hermitian analogues to the Watson transformations for quadratic $\mathbb{Z}$-lattices.

Definition 4.1. [1, Definition 3.1] Let L be a Hermitian lattice over E. For any $n \in \mathbb{N}$, let

$$\Lambda_n(L) = \{x \in L : h(x + y) \equiv h(y) \pmod{n}, \text{ for all } y \in L\},$$

and for any prime p, let

$$\Lambda_n(L_p) = \{x \in L_p : h(x+y) \equiv h(y) \pmod{n}, \text{ for all } y \in L_p\}.$$

The following basic properties of the Λ_p-transformations can be verified in a straightforward manner.

Lemma 4.2. *[1, Lemma 3.2] Let n be a positive integer and p be a prime. Then*

(**a**) *$\Lambda_n(L)$ is a sublattice of L and $\Lambda_n(L_p)$ is a sublattice of L_p.*
(**b**) *$(\Lambda_n(L))_p = \Lambda_n(L_p)$.*
(**c**) *$\Lambda_n(L_p) = L_p$ whenever $p \nmid n$.*
(**d**) *$\mathfrak{n}(\Lambda_n(L)) \subseteq n\mathcal{O}$ and $\mathfrak{n}(\Lambda_n(L_p)) \subseteq n\mathcal{O}_p$.*
(**e**) *If $\mathfrak{P}$ is a prime ideal of $\mathcal{O}$ which is above p, then $\mathfrak{P}L \subseteq \Lambda_p(L)$ and $\mathfrak{P}L_p \subseteq \Lambda_p(L_p)$.*
(**f**) *If N_p splits L_p and $\mathfrak{n}N_p \subseteq p\mathcal{O}_p$, then $N_p \subseteq \Lambda_p(L_p)$.*
(**g**) *$\Lambda_p(L) \subseteq \{x \in L : h(x) \equiv 0 \pmod{p}\}$ and $\Lambda_p(L_p) \subseteq \{x \in L_p : h(x) \equiv 0 \pmod{p}\}$.*

We denote by $\lambda_n(L)$ the primitive lattice obtained from $\Lambda_n(L)$ by scaling the Hermitian map by a suitable rational number. We refer to λ_n as the *Watson transformation* at n.

From now on, unless otherwise stated, we always assume that L *is a primitive positive definite integral ternary Hermitian lattice over E.*

For the Jordan decomposition of Hermitian lattices over E_p, we can refer to [14, Proposition 3.2] for the split case and [8, Sections 7–9] for the non-split case. Note that subnormal $\mathcal{O}_p$-lattices exist only when p is ramified in E. Otherwise L_p is normal and it has an orthogonal basis; so in this case we can always assume that $L_p \cong \langle a\rangle \perp \langle p^i b\rangle \perp \langle p^j c\rangle$ where $a, b, c \in \mathbb{Z}_p^\times$ and $0 \le i \le j$. The following lemma discusses the universality and strict universality of the $\mathcal{O}_p$-lattice L_p.

Lemma 4.3. *Let L_p be a ternary Hermitian lattice over E_p.*

(**a**) *When p is split, then L_p is strictly universal.*
(**b**) *When p is inert, then L_p is strictly universal if and only if $0 \le i \le 1$.*
(**c**) *When p is unramified, L_p is strictly universal if and only if it is universal.*
(**d**) *When p is ramified, if L_p is subnormal or L_p is normal and $i = 0$, then L_p is strictly universal.*

Proof. It suffices to show that L_p has an orthogonal component which is universal, then L_p is strictly universal by Lemma 3.4.

(**a**) Suppose p is split. Then L_p is normal and $\mathbf{N}(\mathcal{O}_p) = \mathbb{Z}_p$. Thus $\langle a\rangle$ is universal.

(**b**) Suppose p is inert. Then L_p is normal and $\mathbf{N}(\mathcal{O}_p^\times) = \mathbb{Z}_p^\times$. When $0 \le i \le 1$, it is easy to see that $\langle a\rangle \perp \langle p^i b\rangle$ is universal. When $2 \le i \le j$, L_p cannot represent the prime elements in $\mathcal{O}_p$, so that L_p is not universal.

(**c**) This is clear from the proofs of (**a**) and (**b**).

(**d**) Suppose p is ramified. If L_p is subnormal or L_p is normal and $i = 0$, then L_p always has a primitive binary modular orthogonal component which is universal by [9, Theorems 1 and 2]. □

Lemma 4.4. *Let L_p be a ternary Hermitian lattice over E_p. Suppose that L_p is not universal. Then*

$$\Lambda_p(L_p) = \{x \in L_p : h(x) \equiv 0 \pmod{p}\}.$$

Proof. The hypothesis implies that $L_p \cong \langle a\rangle \perp \langle p^i b\rangle \perp \langle p^j c\rangle$ where $a, b, c \in \mathbb{Z}_p^\times$. If p is ramified then $1 \le i \le j$; however, if p is inert then $2 \le i \le j$. By Lemma 4.2(**g**), we have $\Lambda_p(L_p) \subseteq \{x \in L_p : h(x) \equiv 0 \pmod{p}\}$. It suffices to show the other direction.

If $x = \alpha x_1 + \beta x_2 + \gamma x_3$ for some $\alpha, \beta, \gamma \in \mathcal{O}_p$, then $h(x) = \mathbf{N}(\alpha)a + \mathbf{N}(\beta)p^i b + \mathbf{N}(\gamma)p^j c$. Thus, $\{x \in L_p : h(x) \equiv 0 \pmod{p}\} = \mathfrak{P}x_1 + \mathcal{O}_p x_2 + \mathcal{O}_p x_3$, where $\mathfrak{P}$ is the maximal ideal of $\mathcal{O}_p$. Following from Lemma 4.2 (**e**) and (**f**), we can conclude that $\mathfrak{P}x_1 + \mathcal{O}_p x_2 + \mathcal{O}_p x_3 \subseteq \Lambda_p(L_p)$. Hence, $\{x \in L_p : h(x) \equiv 0 \pmod{p}\} \subseteq \Lambda_p(L_p)$, and the lemma follows immediately. □

Following from the above lemma, we can conclude that when p is inert and $2 \le i \le j$, we have $\Lambda_p(L_p) = \mathfrak{P}x_1 + \mathcal{O}_p x_2 + \mathcal{O}_p x_3 \cong \langle p^2 a\rangle \perp \langle p^i b\rangle \perp \langle p^j c\rangle$, and therefore $\lambda_p(L_p) = \Lambda_p(L_p)^{1/p^2} \cong \langle a\rangle \perp \langle p^{i-2} b\rangle \perp \langle p^{j-2} c\rangle$. Repeating this procedure finitely many times we obtain a lattice $\lambda_{(p)}(L_p)$ such that

$$\lambda_{(p)}(L_p) \cong \langle a\rangle \perp \langle p^i b\rangle \perp \langle p^j c\rangle,$$

where $0 \le i \le 1$. When p is ramified, L_p is normal, and $1 \le i \le j$, we can apply the similar procedure and obtain a lattice $\lambda_{(p)}(L_p)$ of the form

$$\lambda_{(p)}(L_p) \cong \langle \eta\rangle \perp \langle b\rangle \perp \langle p^j c\rangle,$$

where $\eta \in \mathbb{Z}_p^\times$.

Therefore, if L_p is not universal, we can always obtain a strictly universal lattice by applying the Watson transformation λ_p to L_p a finite number of times. We define $\Delta(L)$ to be the lattice obtained by applying $\lambda_{(p)}$ to L at all primes p where L_p is not universal. Then $\Delta(L)$ is universal at every prime p. We define $\overline{\Delta}(L)$ to be the lattice obtained by applying $\lambda_{(p)}$ to L at all inert primes where L_p is not strictly universal.

Lemma 4.5. *Let K_p be a lattice in a ternary Hermitian space V_p over E_p and $\sigma_p \in U(V_p)$. Then $\Lambda_p(\sigma_p(K_p)) = \sigma_p(\Lambda_p(K_p))$.*

Proof. Let $x \in \Lambda_p(\sigma_p(K_p))$ and $x' \in K_p$ such that $x = \sigma_p(x')$. Then, for any $y' \in K_p$, we have

$$h(x'+y') \equiv h(\sigma_p x' + \sigma_p y') \equiv h(x + \sigma_p y') \equiv h(\sigma_p y') \equiv h(y') \pmod{p}.$$

By definition, $x' \in \Lambda_p(K_p)$ and hence $x \in \sigma_p(\Lambda_p(K_p))$.

Conversely, let $x \in \sigma_p(\Lambda_p(K_p))$. There is an $x' \in \Lambda_p(K_p)$ such that $x = \sigma_p(x')$. Then, for any $y' \in K_p$, we have

$$h(x + \sigma_p y') \equiv h(\sigma_p x' + \sigma_p y') \equiv h(x'+y') \equiv h(y') \equiv h(\sigma_p y') \pmod{p}.$$

Thus $x \in \Lambda_p(\sigma_p(K_p))$. □

Proposition 4.6. *Let L be a regular ternary Hermitian lattice over E and p be a prime. If L_p is not universal, then $\Lambda_p(L)$ is regular.*

Proof. Suppose that $a \longrightarrow (\Lambda_p(L))_q$ for every prime q. We want to show that $a \longrightarrow \Lambda_p(L)$. Since $a \longrightarrow L_q$ for every prime q, by regularity of L, a is represented by L. Hence there is a vector $v \in L$ such that $h(v) = a$. Then $h(v) \equiv 0 \pmod{p}$ because $h(v) \longrightarrow (\Lambda_p(L))_p$. By Lemma 4.4, $v \in (\Lambda_p(L))_p$ which implies that $v \in \Lambda_p(L)$. Therefore $a \longrightarrow \Lambda_p(L)$ and hence $\Lambda_p(L)$ is regular. □

Proposition 4.7. *Let L be a strictly regular ternary Hermitian lattice over E and let p be an inert prime. If L_p is not strictly universal, then $\Lambda_p(L)$ is strictly regular.*

Proof. Suppose that $a \xrightarrow{*} (\Lambda_p(L))_q$ for every prime q. We wish to show that $a \xrightarrow{*} \Lambda_p(L)$. Since, by Hasse's Principle, a is represented by EL, there exists $v \in EL$ such that $h(v) = a$. Let N be the lattice $\mathcal{O}v$. Then it is obvious that $N_q \xrightarrow{*} (\Lambda_p(L))_q$ for every prime q. When $q \neq p$, $(\Lambda_p(L))_q = \Lambda_p(L_q) = L_q$, so that $N_q \xrightarrow{*} L_q$. Since L_p is not strictly universal, L_p is not universal by Lemma 4.3(c). By Lemma 4.4, we know that $(\Lambda_p(L))_p = \Lambda_p(L_p) = \{x \in L_p : h(x) \equiv 0 \pmod{p}\}$. As $N_p \xrightarrow{*} \Lambda_p(L_p)$, there is a primitive sublattice H_p of $\Lambda_p(L_p)$ such that $\sigma_p H_p = N_p$ for some $\sigma_p \in U((EL)_p)$. We define a lattice G on EN by

$$G_q = \begin{cases} N_q & \text{if } q \neq p; \\ \sigma_p(E_p H_p \cap L_p) & \text{if } q = p. \end{cases}$$

Note that $N \subseteq G$ because $N_q \subseteq G_q$ at every prime q. Additionally, since p is inert and N is free, G is also a free unary lattice.

We claim that $\Lambda_p(G_p) = \sigma_p(E_p H_p \cap \Lambda_p(L_p))$. By Lemma 4.5, we have that $\Lambda_p(G_p) = \sigma_p(\Lambda_p(E_p H_p \cap L_p))$. Now we want to show that $\Lambda_p(E_p H_p \cap L_p) = E_p H_p \cap \Lambda_p(L_p)$. Let $x \in \Lambda_p(E_p H_p \cap L_p)$. Then $x \in E_p H_p$ and $x \in L_p$ such that $h(x) \equiv 0 \pmod{p}$. Thus $x \in E_p H_p \cap \Lambda_p(L_p)$ by Lemma 4.4. Conversely, let $x \in E_p H_p \cap \Lambda_p(L_p)$. Then $x \in E_p H_p \cap L_p$ such that $h(x+y) \equiv h(y) \pmod{p}$ for all $y \in L_p$. Since $E_p H_p \cap L_p \subseteq L_p$, it is obvious that $h(x+y) \equiv h(y) \pmod{p}$ for all $y \in E_p H_p \cap L_p$. Thus $x \in \Lambda_p(E_p H_p \cap L_p)$. Hence $\Lambda_p(E_p H_p \cap L_p) = E_p H_p \cap \Lambda_p(L_p)$ and $\Lambda_p(G_p) = \sigma_p(E_p H_p \cap \Lambda_p(L_p))$.

Since H_p is primitive in $\Lambda_p(L_p)$, we know that $E_pH_p \cap \Lambda_p(L_p) = H_p$. Then $\Lambda_p(G_p) = \sigma_p(H_p) = N_p$. For $q \neq p$, we have $\Lambda_p(G_q) = G_q = N_q$. Therefore, $\Lambda_p(G) = N$.

We know that $E_pH_p \cap L_p$ is primitively contained in L_p; so $G_p \xrightarrow{*} L_p$. Since $G_q = N_q \xrightarrow{*} L_q$ for $q \neq p$, by strict regularity of L, $G \xrightarrow{*} L$. We claim that $N = \Lambda_p(G) \xrightarrow{*} \Lambda_p(L)$. Let $\sigma : G \xrightarrow{*} L$ be a representation such that $\sigma(G)$ is primitive in L. Note that for any $z \in \sigma(\Lambda_p(G))$, $h(z) \equiv 0 \pmod{p}$ and $z \in L$, so that $z \in \Lambda_p(L)$. Suppose that x is an element in $\Lambda_p(L)$ such that $\alpha x \in \sigma(\Lambda_p(G))$ for some nonzero element $\alpha \in \mathcal{O}$. Then $x \in \sigma(G)$ because $L/\sigma(G)$ is torsion free and $\sigma(\Lambda_p(G)) \subseteq \sigma(G)$. Additionally, $x \in \Lambda_p(L)$ implies that $h(x + y) \equiv h(y) \pmod{p}$ for all $y \in L$. Since $\sigma(G) \subseteq L$, we have that $h(x + y) \equiv h(y) \pmod{p}$ for all $y \in \sigma(G)$. Hence $x \in \Lambda_p(\sigma(G)) = \sigma(\Lambda_p(G))$, by Lemma 4.5. Then $\Lambda_p(L)/\sigma(\Lambda_p(G))$ is torsion free, which means that $\Lambda_p(G) \xrightarrow{*} \Lambda_p(L)$. □

5. Proof of results

For the remainder of the paper we fix an imaginary quadratic field $E = \mathbb{Q}(\sqrt{-m})$. A positive number is said to be bounded if it is bounded above by a constant depending only on E. An inequality of the form $A \ll B$ will mean that there exists a constant k such that $|A| < kB$.

Proposition 5.1. *Let L be a strictly regular ternary Hermitian lattice over E and let r be a bounded rational prime. If L_q is strictly universal for all $q \nmid rD_E$, then δ_L is bounded.*

Proof. Let L be such a lattice. Since $\delta_L \leq \mu_1(L)\mu_2(L)\mu_3(L)$, it suffices to bound these three successive minima of L. Using Dirichlet's Theorem, choose $\ell_1 < \ell_2$ to be two bounded primes such that $\ell_1 \xrightarrow{*} L_q$ and $\ell_2 \xrightarrow{*} L_q$ for all $q \mid rD_E$. By strict regularity of L, ℓ_1 and ℓ_2 are primitively represented by L. Thus $\mu_1(L) \leq \ell_1$. Let z_1, z_2 be two primitive vectors in L such that $h(z_i) = \ell_i$ for $1 \leq i \leq 2$. If $z_2 = \gamma z_1$ for some $\gamma \in E$, then $\gamma \in \mathcal{O}$ and $\ell_2 = \mathbf{N}(\gamma)\ell_1$. This is a contradiction, so $\mu_2(L) \leq \ell_2$.

Consider the leading binary sublattice $B = \mathrm{span}_E\{z_1, z_2\} \cap L$. By Lemma 3.3, there exists some prime p and positive integer $t = t(B, p)$ such that B_p does not primitively represent any p-adic integer in $p^t\mathbb{Z}_p$. But we claim that there is always an element d in $p^t\mathbb{Z}_p$ such that $d \xrightarrow{*} L_p$. If L_p is strictly universal, then $d = p^t \xrightarrow{*} L_p$. Otherwise $L_p \cong \langle a\rangle \perp \langle p^i b\rangle \perp \langle p^j c\rangle$ where $a, b, c \in \mathbb{Z}_p^\times \cap \mathbb{Z}$ are bounded and $i \geq 1$. Since $B \xrightarrow{*} L$ and $\delta_B \leq \ell_1\ell_2$, it must be the case that i is bounded. If $i \geq t$, then $d = p^i b \xrightarrow{*} L_p$. If $i < t$, then $\langle a\rangle \perp \langle p^i b\rangle$ primitively represents every element in $p^t\mathbb{Z}_p$. Therefore, $d = p^t \xrightarrow{*} L_p$. Using Dirichlet's Theorem, choose $\ell_3 > \ell_2$ such that $\ell_3 d \xrightarrow{*} L_q$ for all $q \mid rD_E$. Then $\ell_3 d \xrightarrow{*} L$ but $\ell_3 d \overset{*}{\not\to} B$.

From the above discussion we see that $\mu_3(L)$ is bounded by a constant depending only on B. Since $\delta_B \leq \ell_1\ell_2$, which depends only on E, the number of possible isometry classes of B is bounded. Therefore, there is a constant C depending only on E such that $\mu_3(L) < C$ and $\delta_L \leq \ell_1\ell_2 C$. □

Proposition 5.2. *Let L be a regular ternary Hermitian lattice over E and p be a fixed rational prime. If L_q is universal for all primes $q \neq p$, then $\mu_1(L) \ll p^{\frac{1}{2}}$ and $\mu_2(L) \ll p$.*

Proof. Let L be such a lattice. If L_p is universal, then L is universal, and it is easy to see that $\mu_1(L) = 1$ and $\mu_2(L)$ is bounded by the smallest inert prime in E. If $p = 2$, then we can use Dirichlet's Theorem as in the proof of Proposition 5.1 to show that $\mu_1(L) \leq \ell_1$ and $\mu_2(L) \leq \ell_2$ for bounded primes ℓ_1 and ℓ_2. In either of those cases, $\mu_1(L) \ll p^{\frac{1}{2}}$ and $\mu_2(L) \ll p$. Otherwise, we can assume that $p > 2$, L_p is normal, and $L_p \cong \langle a\rangle \perp \langle p^i b\rangle \perp \langle p^j c\rangle$. Let $\mathcal{F} = \left\{\alpha \in \mathbb{Z}^+ : \left(\frac{a}{p}\right) = \left(\frac{\alpha}{p}\right)\right\}$, so L_p represents all elements in $\mathcal{F}$, and subsequently L also represents all elements in $\mathcal{F}$. For a positive integer G we consider $S(G) = \{\alpha \in \mathcal{F} : 1 \leq \alpha \leq G\}$, and let $g = \min\{G : S(G) \neq \emptyset\}$. Then by [3, Corollary 3.3] we have that $g \ll p^{\frac{1}{2}}$. Thus $\mu_1(L) \ll p^{\frac{1}{2}}$.

Let ℓ be the smallest prime such that $\left(\frac{-m}{\ell}\right) = -1$ and $\ell \nmid 2p\mu_1(L)$. Using [3, Corollary 3.3] with character of modulus $8m$ and desired value -1 we get that $\ell \ll p^{\frac{1}{2}}$. Let t be the smallest positive integer such that $\gcd(t, 2p\ell m) = 1$ and $\left(\frac{t}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{\ell}{p}\right)$. Then

$$\left(\frac{\ell t}{p}\right) = \left(\frac{\ell}{p}\right)\left(\frac{t}{p}\right) = \left(\frac{a}{p}\right).$$

Thus $\ell t \longrightarrow L_p$ and since L_r is universal for all other primes r we know that $\ell t \longrightarrow L$. By [3, Corollary 3.3] we see that $t \ll p^{\frac{1}{2}}$; so $\ell t \ll p$. Consider the one dimensional Hermitian spaces $[\mu_1(L)]$ and $[\ell t]$, and assume that they are isometric. Then the associated binary quadratic spaces $[\mu_1(L), \mu_1(L)m]$ and $[\ell t, \ell t m]$ are isometric and have the same ℓ-adic Hasse invariant. However, $S_\ell([\mu_1(L), \mu_1(L)m]) = (\mu_1(L), -m) = 1$ and $S_\ell([\ell t, \ell t m]) = (\ell t, -m) = (t, -m)(\ell, -m) = -1$. Hence $[\mu_1(L)]$ and $[\ell t]$ cannot be isometric, which means that $\ell t \notin \mu_1(L)\mathbf{N}(E)$. Therefore $\mu_2(L) \ll p$. □

Proposition 5.3. *The prime divisors of δ_L are bounded for all strictly regular ternary Hermitian lattices L over E.*

Proof. If a strictly regular ternary Hermitian lattice over E is strictly universal for all $q \nmid rD_E$ where r is a bounded prime, then by Proposition 5.1, the discriminant of this lattice is bounded. Recall that $\overline{\Delta}(L)$ is the strictly regular lattice obtained by applying $\lambda_{(p)}$ to L at all inert primes where L_p is not strictly universal. Thus $\overline{\Delta}$ satisfies the previous condition. Therefore, given a strictly regular ternary Hermitian lattice L over E and a prime divisor p of $\delta_{\overline{\Delta}(L)}$, p must be bounded.

We claim that if $p \mid \delta_L$ but $p \nmid \delta_{\overline{\Delta}(L)}$, then p is also bounded. We can assume that $p > 2$. By the definition of $\overline{\Delta}(L)$, such p must be inert. Thus we can assume that $L_p \cong \langle a\rangle \perp \langle p^{2t} b\rangle \perp \langle p^{2t} c\rangle$, where $a, b, c \in \mathbb{Z}_p^\times$ and $t > 0$. In this case, for all other inert primes $q \neq p$, we can apply the λ_q-transformation repeatedly and hence assume that L_q is strictly universal for all primes $q \nmid pD_E$. Let S be the product of prime divisors q of pD_E where L_q is not strictly universal. For all $q \mid S$, $L_q \cong \langle a_q\rangle \perp L'_q$.

Let $\mathcal{F} = \left\{\alpha \in \mathbb{Z}^+ : \left(\frac{a_q}{q}\right) = \left(\frac{\alpha}{q}\right) \text{ for all } q \mid S\right\}$. By strict regularity, $\alpha \xrightarrow{*} L$ for every $\alpha \in \mathcal{F}$. Following from the proof of Proposition 5.2, $\mu_1(L) \ll p^{\frac{1}{2}}$. Let ℓ be the smallest prime such that $\left(\frac{-m}{\ell}\right) = -1$ and $\ell \nmid 2S\mu_1(L)$. Let t be the smallest positive integer such that $\gcd(t, 2S\ell m) = 1$ and $\left(\frac{t}{q}\right) = \left(\frac{a_q}{q}\right)\left(\frac{\ell}{q}\right)$ for all $q \mid S$. Again following from the proof of Proposition 5.2, $\mu_2(L) \ll p$.

Let B be a leading binary component of L; so $p^2 \mid \delta_B$. Then $p^2 \leq \delta_B \leq \mu_1\mu_2 \ll p^{\frac{3}{2}}$. Hence $p \ll 1$, that is, p is bounded. □

Proposition 5.4. *Let E be an imaginary quadratic field that does not support a primitive regular binary Hermitian lattice. Then the prime divisors of δ_L are bounded for all regular ternary Hermitian lattices L over E.*

Proof. First we claim that if L is a universal ternary Hermitian lattice over E, then δ_L is bounded. Since L is universal, $\mu_1(L) = 1$ and $\mu_2(L) \leq \ell$ where ℓ is the smallest inert prime in E. For the third successive minimum, let B be a leading binary section of L. Then B cannot be regular, and there must exist a positive integer a such that $a \longrightarrow B_q$ for every prime q, but $a \not\longrightarrow B$. Note that this integer a can be chosen to be bounded. However, it is clear that $a \longrightarrow L$ because L is universal. Thus $\mu_3(L) \leq a$ and δ_L is bounded.

Now, suppose that L is a regular ternary Hermitian lattice over E. Since $\Delta(L)$ is universal, if $p \mid \delta_{\Delta(L)}$ then p is bounded. When $p \mid \delta_L$ but $p \nmid \delta_{\Delta(L)}$, p is either inert or ramified, and in the latter case p is bounded. Hence we can assume p is inert and $L_p \cong \langle a\rangle \perp \langle p^{2t}b\rangle \perp \langle p^{2t}c\rangle$ with $t > 0$. Furthermore, we can apply the λ_q-transformation for $q \neq p$ and assume that L_q is universal. By Proposition 5.2, $\mu_1(L) \ll p^{\frac{1}{2}}$ and $\mu_2(L) \ll p$. Let B be a leading binary component of L; so $p^2 \mid \delta_B$. Then $p^2 \leq \delta_B \leq \mu_1\mu_2 \ll p^{\frac{3}{2}}$. Hence $p \ll 1$, that is, p is bounded. □

Proof of Theorem 1.1. Suppose that E is an imaginary quadratic field that supports a primitive regular binary Hermitian lattice, K. We can apply the Watson transformation to K and get a primitive *universal* binary Hermitian lattice, $\Delta(K)$. Then we can obtain an infinite family of non-isometric primitive regular ternary Hermitian lattices $\{\Delta(K) \perp \langle a\rangle : a \in \mathbb{Z}^+\}$.

Now suppose E is an imaginary quadratic field that does not support any primitive regular binary Hermitian lattice. Let L be a regular ternary Hermitian lattice over E and fix a prime divisor p of δ_L. Note that p is bounded by Proposition 5.4. We claim that $\mathrm{ord}_p(\delta_L)$ is bounded. We can apply the λ_q-transformation at primes $q \neq p$ since they do not change $\mathrm{ord}_p(\delta_L)$. Thus we can assume that L_q is universal for $q \neq p$. Then by Proposition 5.2 we know that the first two successive minima of L are bounded. Let B be a leading binary section of L. Then B cannot be regular, and there must exist a bounded positive integer a such that $a \longrightarrow B_q$ for every prime q, but $a \not\longrightarrow B$. However, it is clear that $a \longrightarrow L_q$ at every prime, and by regularity of L, $a \longrightarrow L$. Thus $\mu_3(L) \leq a$ and hence $\mathrm{ord}_p(\delta_L)$ is bounded. Therefore, δ_L itself must be bounded and hence there must be only finitely many isometry classes of primitive regular ternary Hermitian lattices over E. □

Proof of Theorem 1.2. Towards a contradiction, suppose that $\mathcal{S}$ is a set of infinitely many non-isometric primitive strictly regular ternary Hermitian lattices over E. By Proposition 5.3 we know that the prime divisors of δ_L for all L in $\mathcal{S}$ come from a fixed finite set. Thus there must be some bounded prime p such that $\{\mathrm{ord}_p(\delta_L) : L \in \mathcal{S}\}$ is an unbounded set. At $q \nmid pD_E$, the Watson transformation λ_q does not change $\mathrm{ord}_p(\delta_L)$. Therefore, for any $L \in \mathcal{S}$ we may assume that L_q is strictly universal for all $q \nmid pD_E$. Then for any $L \in \mathcal{S}$, δ_L is bounded by Proposition 5.1. This is a contradiction. □

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